Algorithms for Square Root Extraction

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Contents

1 Greek Method

$$
\begin{aligned}
a_1 &= a & a_{n+1} &= \frac{a_n + b_n}{2} \\
b_1 &= 1 & b_{n+1} &= \frac{2a_n b_n}{a_n + b_n} \\
\{a_n\} &\to \sqrt{a} \\
\{b_n\} &\to \sqrt{a}
\end{aligned}
$$

- First terms are 1 and $\sqrt{a}$
- Subsequent terms of $\{a_n\}$ are the average of the previous a and b
- Subsequent terms of $\{a_n\}$ are twice the product divided by the sum

Method to Prove Convergence

1. Prove $a_n b_n = a \;\forall\, n \in \mathbb{N}$
2. Prove $a_n < b_n \;\forall\, n \in \mathbb{N}$
3. Prove a_n is strictly decreasing
4. Prove b_n is strictly increasing
5. Prove $\lim_{n\to\infty} a_n^2 = \lim_{n\to\infty} b_n^2 = a$

1.1 Product of Terms

Base Case

$n = 1$:

$$a_1 b_1 = a \times 1 = a$$

Induction Step

Assume $a_k b_k = a$, then

$$a_{k+1} b_{k+1} = \left(\frac{a_k + b_k}{2}\right)\left(\frac{2a_k b_k}{a_k + b_k}\right) = a_k b_k = a$$

1.2 $\{a_n\} > \{b_n\}$

- $\forall\, n \in \mathbb{N},\ a_n \geq b_n$

Base Case for Induction

$n = 1$:

$$a_1 = a > b_1 = 1$$

Induction Step

Assume $a_k > b_k$, then

$$(a_k - b_k)^2 > 0$$

$$a_k^2 - 2a_k b_k + b_k^2 > 0$$

$$a_k^2 - 2a_k b_k + b_k^2 > 0$$

$$a_k^2 + 2a_k b_k + b_k^2 > 4a_k b_k$$

$$(a_k + b_k)^2 > 4ab_k$$

$$\frac{a_k + b_k}{2} > \frac{2a_k b_k}{a_k + b_k}$$

$$a_{k+1} \quad > \quad b_{k+1}$$

- By induction, $\{a_n\} > \{b_n\}$

1.3 $\{a_n\}$ is Strictly Decreasing

- Since $a_n > b_n \ \forall n \in \mathbb{N}$:

$$\begin{aligned} a_n &> b_n \\ 2a_n &> a_n + b_n \\ a_n &> \frac{a_n + b_n}{2} \\ a_n &> a_{n+1} \end{aligned}$$

1.4 $\{b_n\}$ is Strictly Increasing

- Since $a_n > b_n \ \forall n \in \mathbb{N}$:

$$\begin{aligned} b_n &< a_n \\ b_n^2 &< a_n b_n \\ a_n b_n + b_n^2 &< 2a_n b_n \\ \\ b_n\left(a_n + b_n\right) &< 2a_n b_n \\ \\ b_n &< \frac{2a_n b_n}{a_n + b_n} \\ \\ b_n &< b_{n+1} \end{aligned}$$

1.5 The Sequences $\{a_n\}$ and $\{b_n\}$ Converge to $\sqrt{a}$

- $\forall\, \varepsilon > 0,\ \exists, N_1 \in \mathbb{N}$ s.t. $\forall\, n, m > N_1,\ |a_n - a_m| < \frac{1}{2}\varepsilon$
- $m > n \Rightarrow a_m < a_n \Rightarrow |a_n - a_m| = a_n - a_m$

$$\begin{aligned}
\frac{1}{2}\varepsilon &> a_n - a_{n+1} \\
\frac{1}{2}\varepsilon &> a_n - \frac{a_n + b_n}{2} \\
\frac{1}{4}\varepsilon^2 &> \left(\frac{a_n - b_n}{2}\right)^2 \\
\frac{1}{4}\varepsilon^2 &> \frac{a_n^2 - 2a_nb_n + b_n^2}{4} \\
\varepsilon^2 &> a_n^2 - 2a + b_n^2 \\
\varepsilon^2 &> (a_n - b_n)^2 \\
\varepsilon &> a_n - b_n \\
\lim_{n\to\infty} a_n &= \lim_{n\to\infty} b_n
\end{aligned}$$

2 Newton's Method

$$1 \leq b_1 \leq a$$

$$b_{n+1} = \frac{b_n + {}^{a}/_{b_n}}{2}$$

$$\{b_n\} \rightarrow \sqrt{a}$$

- First term is any guess between 1 and a
- Subsequent terms are the average of the prior term and a divided by the prior term
- Also known as the Babylonian Method or Heron's Method

2.1 $b_1 = \sqrt{a}$

- For the trivial case where $b_1 = \sqrt{a}$, the entire sequence is identically $\sqrt{a}$ as shown by this induction step:

$$\text{Assume } b_k = \sqrt{a}$$

$$b_{k+1} = \frac{b_k + {}^{a}\!/_{b_k}}{2} = \frac{\sqrt{a} + {}^{a}\!/_{\sqrt{a}}}{2} = \frac{\sqrt{a} + \sqrt{a}}{2} = \sqrt{a}$$

2.2 The Tail of $\{b_n\}$ is Bounded Below

- Let $\delta_n = |\sqrt{a} - b_n|$. Then for any b_n, we have:

$$\begin{aligned}
\delta_n &\geq 0 \\
\delta_n^2 &\geq 0 \\
\delta_n^2 + 2a - 2\sqrt{a}\delta &\geq 2a - 2\sqrt{a}\delta_n \\
a - 2\sqrt{a}\delta + \delta_n^2 + a &\geq 2\sqrt{a}\left(\sqrt{a} - \delta_n\right) \\
\left(\sqrt{a} - \delta_n\right)^2 + a &\geq 2\sqrt{a}\left(\sqrt{a} - \delta_n\right) \\
\frac{\sqrt{a} - \delta_n + \frac{a}{\sqrt{a}-\delta_n}}{2} &\geq \sqrt{a} \\
\frac{b_n + {}^a/_{b_n}}{2} &\geq \sqrt{a} \\
b_{n+1} &\geq \sqrt{a}
\end{aligned}$$

- The tail of the sequence is bounded below by $\sqrt{a}$

2.3 The Tail of $\{b_n\}$ is Monotone Decreasing

- Since $b_n \geq \sqrt{a}\ \forall\, n > 1$,

$$\begin{aligned} b_n &\geq \sqrt{a} \\ b_n\sqrt{a} &\geq a \\ \sqrt{a} &\geq \frac{a}{b_n} \\ \therefore b_n &\geq \frac{a}{b_n} \quad \forall\, n > 1 \end{aligned}$$

$$\begin{aligned} 2b_n &\geq b_n + \frac{a}{b_n} \\ b_n &\geq \frac{b_n + {}^{a}\!/_{b_n}}{2} \\ b_n &\geq b_{n+1} \quad \forall\, n > 1 \end{aligned}$$

- Thus the tail of the sequence is monotone decreasing.

2.4 $\{b_n\}$ Converges to $\sqrt{a}$

- $\forall\, \varepsilon > 0,\ \exists,\ N \in \mathbb{N}$ s.t. $\forall\, n, m > N,\ |b_n - b_m| < \frac{1}{2a}\varepsilon.$
- $m > n \Rightarrow b_m \leq b_n \Rightarrow |b_n - b_m| = b_n - b_m.$

$$\begin{aligned}
\frac{1}{2a}\varepsilon &> b_n - b_{n+1} \\
\frac{1}{2a}\varepsilon &> \frac{2b_n}{2} - \frac{b_n + {}^{a}\!/_{b_n}}{2} \\
\frac{1}{2a}\varepsilon &> \frac{{}^{a}\!/_{b_n} - b_n}{2} \\
\varepsilon &> \frac{a}{b_n}\left(a - b_n^2\right) \geq a - b_n^2 \\
\varepsilon &> a - b_n^2 \\
\lim_{n\to\infty} b_n^2 &= a \\
\\
\lim_{n\to\infty} b_n &= \sqrt{a}
\end{aligned}$$

3 Bahkshali Method

$$\sqrt{a} = \sqrt{b^2 + d} \approx b + \frac{d}{2b} - \frac{\left(d/2b\right)^2}{2\left(b + \frac{d}{2b}\right)}$$

- To find $\sqrt{a}$, one must first guess at b and then determine d by the relation $d = a - b^2$
- The approximate square root is then given by the formula
- To iterate for a more accurate solution, a sequence can be defined as:

$$\begin{aligned} d_n &= a - b_n^2 \\ b_{n+1} &= b_n + \frac{d_n}{2b_n} - \frac{\left(\frac{d_n}{2b_n}\right)^2}{2\left(b_n + \frac{d_n}{2b_n}\right)} \end{aligned}$$

Rewriting the Sequence

$$\begin{aligned}
b_{n+1} &= b_n + \frac{d_n}{2b_n} - \frac{\left(\frac{d_n}{2b_n}\right)^2}{2\left(b_n + \frac{d_n}{2b_n}\right)} \\
&= b_n + \frac{d_n}{2b_n} - \frac{d_n^2}{8b_n^3 + 4b_n d_n} \\
&= b_n + \frac{a - b_n^2}{2b_n} - \frac{\left(a - b_n^2\right)^2}{8b_n^3 + 4b_n\left(a - b_n^2\right)} \\
&= b_n + \frac{a - b_n^2}{2b_n} - \frac{a^2 - 2ab_n^2 + b_n^4}{4b_n^3 + 4ab_n} \\
&= b_n + \frac{a - b_n^2}{2b_n} - \frac{a^2 - 2ab_n^2 + b_n^4}{4b_n\left(b_n^2 + a\right)} \\
&= \frac{4b_n^2\left(b_n^2 + a\right) + 2\left(a - b_n^2\right)\left(b_n^2 + a\right) - \left(a - b_n^2\right)^2}{4b_n^3 + 4ab_n} \\
b_{n+1} &= \frac{a^2 + 6ab_n^2 + b_n^4}{4b_n^3 + 4ab_n}
\end{aligned}$$

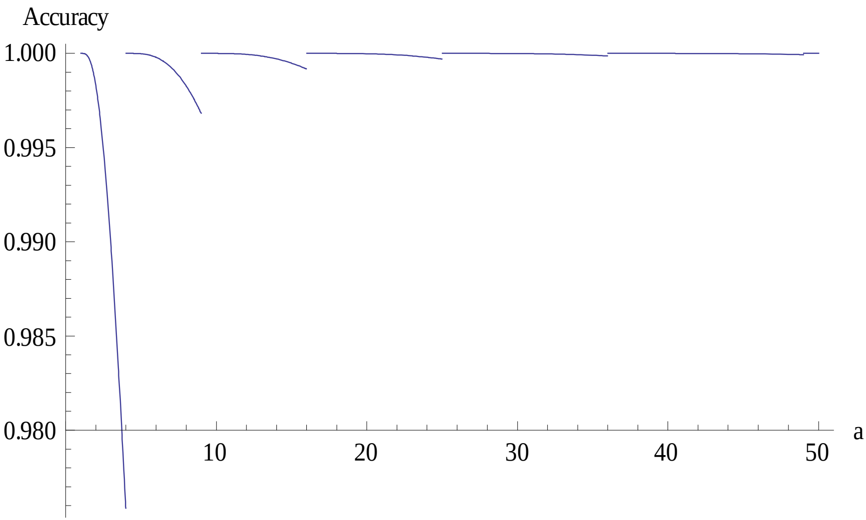
Newton's Method Revisited

$$b_{n+2} = \frac{b_{n+1} + {}^{a}\!/_{b_{n+1}}}{2} = \frac{\frac{b_n + {}^{a}\!/_{b_n}}{2} + \frac{a}{\frac{b_n + {}^{a}\!/_{b_n}}{2}}}{2}$$

$$\begin{aligned}
b_{n+2} &= \frac{b_n + {}^{a}\!/_{b_n} + \frac{4a}{b_n + {}^{a}\!/_{b_n}}}{4} \\
&= \frac{b_n}{4} + \frac{a}{4b_n} + \frac{ab_n}{b_n^2 + a} \\
&= \frac{b_n^2\left(b_n^2 + a\right) + a\left(b_n^2 + a\right) + 4ab_n^2}{4b_n\left(b_n^2 + a\right)} \\
b_{n+2} &= \frac{a^2 + 6ab_n^2 + b_n^4}{4b_n^3 + 4ab_n}
\end{aligned}$$

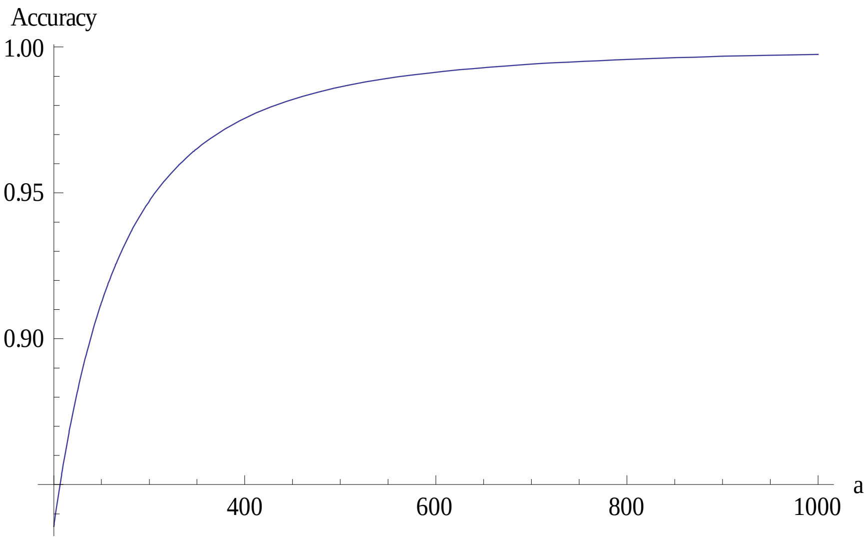
- The Bahkshali formula is simply two iterations of Newton's method

Efficiency



$$b = \lfloor \sqrt{a} \rfloor$$

Efficiency



$$b = \sqrt{a} - 10$$

4 High School Method

- Starting at decimal point, digits are split into pairs
- First digit of root is the largest integer whose square is less than the first pair of digits

	3 2. 4
	10 49. 76
3	9
	1 49
62	1 24
	25 76
644	25 76
	0

4.1 Procedure

1. Divide the number, a, into pairs of digits, positioned to keep the decimal point between pairs.

2. The first digit, d_1, of the root, b, is the largest integer whose square is less than the first pair of digits.

3. Square the first digit and write it below the first pair of digits, then subtract.

4. Bring down the next pair of digits to complete the difference, similar to long division.

5. The each subsequent digit, d_i, of the root is the largest integer such that $d_i \left(20' d_1 d_2 \ldots d'_{i-1} + d_2\right)$ is less than or equal to the remaininder.

6. Subtract the product from step 5 from the difference, then bring down the next pair. Zeroes can be appended to the end as in long division.

7. Continue until the desired accuracy is reached or the difference is 0.

4.2 Binomial Expansion

$$
\begin{aligned}
{}'d_1 d_2'^2 &= \left(10d_1 + d_2\right)^2 \\
&= 100d_1^2 + 20d_1 d_2 + d_2^2 \\
&= 100d_1^2 + \left(20d_1 + d_2\right) d_2
\end{aligned}
$$

$$
\begin{aligned}
{}'d_1 d_2 d_3'^2 &= \left(100d_1 + 10d_2 + d_3\right)^2 \\
&= \left(10\left(10d_1 + d_2\right) + d_3\right)^2 \\
&= 100\left(10d_1 + d_2\right)^2 + 20\left(10d_1 + d_2\right) d_3 + d_3^2 \\
&= 100\left(10d_1 + d_2\right)^2 + \left(200d_1 + 20d_2 + d_3\right) d_3 \\
&= 100'd_1 d_2'^2 + \left(20'd_1 d_2' + d_3\right) + d_3
\end{aligned}
$$

Binomial Expansion

$$
\begin{aligned}
{}'d_1d_2d_3d_4'^2 &= \left(1000d_1 + 100d_2 + 10d_3 + d_4\right)^2 \\
&= \left(10\left(100d_1 + 10d_2 + d_3\right) + d_2\right)^2 \\
&= 100\left(100d_1 + 10d_2 + d_3\right)^2 + 20\left(100d_1 + 10d_2 + d_3\right)d_4 + d_4^2 \\
&= 100\left(100d_1 + 10d_2 + d_3\right)^2 + \left(2000d_1 + 200d_2 + 20d_3 + d_4\right)d_4 \\
&= 100'd_1d_2d_3'^2 + \left(20'd_1d_2d_3' + d_4\right) + d_4
\end{aligned}
$$

- The total of the terms subtracted is equal to the square of the digits

$$
\begin{array}{r|r}
 & d_1 \quad d_2. \quad d_3 \\
\hline
 & a_1a_2\ a_3a_4.\ a_5a_6 \\
d_1 & \underline{\text{-}d_1^2} \qquad\qquad \\
 & a_1a_2 - d_1^2\ \ a_3a_4 \qquad \\
20d_1 + d_2 & \underline{-\left(20d_1 + d_2\right)d_2} \qquad \\
 & \ldots \qquad \\
200d_1 + 20d_2 + d_3 & \underline{-\left(200d_1 + 20d_2 + d_3\right)d_3}
\end{array}
$$

4.3 Some Nomenclature

- $k = \lfloor \log_{100} a \rfloor + 1$, i.e. k is such that $100^{k-1} \leq a < 100^k$.
- $'d_i'$ in single quotes will represent the digit and the variable d_i will represent $'d_i' \times 10^{k-i}$
- $b_n = \sum_{i=1}^{n} d_i$. For example, if $k = 3$, $b_3 = d_1 + d_2 + d_3 =' d_1 d_2 d_3'$
- $b_n$is increasing
- Need to show why $b_n$is bounded above by $\sqrt{a}$.

4.4 Proof

- First Digit: $'d_1'$

$$d_1^2 \leq a < \left(d_1 + 10^{k-1}\right)^2$$

- The second inequality ensures that if a higher digit would work, that should be $'d_1'$.

- Subtracting the square of $'d_1'$ from the first pair of digits is identical to subtracting d_1^2 from a.

- The first digits of $a - d_1^2$ will be the same as the remainder term with the next pair of digits brought down in the algorithm.

- In the tabular form, the zeroes are excluded and digits are only brought down as necessary.

Proof

- The next $'d'_n$ is chosen as high as possible so that the product of the new digit and the sum of 20 times the digits already found plus the new digit is less than what is left from the subtraction.

$$d_n \times (2b_{n-1} + d_n) \leq a - b_{n-1}^2 < \left(d_n + 10^{k-n}\right) \times \left(2b_{n-1} + \left(d_n + 10^{k-n}\right)\right)$$

- The difference between b_{n-1} and b_n is d_n, so we can take:

$$\begin{aligned}
b_n - b_{n-1} &= d_n \\
b_n &= d_n + b_{n-1} \\
b_n^2 &= d_n^2 + 2d_n b_{n-1} + b_{n-1}^2 \\
b_n^2 - 2b_{n-1}d_n - d_n^2 &= b_{n-1}^2
\end{aligned}$$

Proof

- Substituting this into our inequality,

$$\begin{aligned} d_n \times (2b_{n-1} + d_n) &\leq a - b_{n-1}^2 \\ 2b_{n-1}d_n + d_n^2 &\leq a - b_n^2 + 2b_{n-1}d_n + d_n^2 \\ b_n^2 &\leq a \\ b_n &\leq \sqrt{a} \end{aligned}$$

- So $'d_n'$ is chosen so that $b_n \leq \sqrt{a}$ and therefore bounded above. If we focus on the other side of the inequality (the strict inequality), we get:

$$\begin{aligned} a - b_{n-1}^2 &< \left(d_n + 10^{k-n}\right) \times \left(2b_{n-1} + \left(d_n + 10^{k-n}\right)\right) \\ a - b_{n-1}^2 &< 2b_{n-1}\left(d_n + 10^{k-n}\right) + \left(d_n + 10^{k-n}\right)^2 \\ a - b_{n-1}^2 &< 2b_{n-1}d_n + 2 \times 10^{k-n}b_{n-1} + d_n^2 + 2 \times 10^{k-n}d_n + 100^{k-n} \\ a - b_n^2 + 2b_{n-1}d_n + d_n^2 &< 2b_{n-1}d_n + 2 \times 10^{k-n}b_{n-1} + d_n^2 + 2 \times 10^{k-n}d_n + 100^{k-n} \\ a - b_n^2 &< 2 \times 10^{k-n}b_{n-1} + 2 \times 10^{k-n}d_n + 100^{k-n} \\ a - b_n^2 &< 10^{k-n}(2b_{n-1} + 2d_n + 10^{k-n}) \\ a - b_n^2 &< 10^{k-n}(2b_n + 10^{k-n}) \end{aligned}$$

Proof

$\{b_n\}$ is bounded above by $\sqrt{a}$, so for any $\varepsilon > 0$, $\exists\, N \in \mathbb{N}$ such that $10^{k-N} < \frac{\varepsilon}{2\sqrt{a}}$, thus $\forall\, n > N$:

$$10^{k-N} < \frac{\varepsilon}{2\sqrt{a}} \leq \frac{\varepsilon}{2b_n} < \frac{\varepsilon}{2b_n + 10^{k-N}}$$

$$10^{k-N}\left(2b_n + 10^{k-N}\right) < \varepsilon$$

$$a - b_n^2 < \varepsilon$$

$$\begin{aligned} \lim_{n\to\infty} b_n^2 &= a \\ \lim_{n\to\infty} b_n &= \sqrt{a} \end{aligned}$$

4.5 Example 2

- For a second example, $\sqrt{1574.5024}$:

	3 9. 6 8
	1 5 7 4.5 0 2 4
3	- 9 0 0.
	6 7 4.5 0 2 4
69	- 6 2 1.
	5 3.5 0 2 4
786	- 4 7.1 6
	6.3 4 2 4
7928	- 6.3 4 2 4
	0.0 0 0 0

For this example, the full expansions were used to demonstrate the equivalence between correctly placing the values under the pairs of digits and accounting for the $\left(10^{k-N}\right)^2$ terms in the binomial expansion.

Algorithms for Square Root Extraction

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